Response of a Laminated Beam to a Moving Load

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A study of the steady-state response of an infinite composite beam subjected to a moving concentrated load and supported by an elastic foundation with and without damping is presented. A rectangular beam, with alternating layers of two different elastic materials is considered, and the equations derived by C. T. Sun, which include microstructure, are employed. Results are also obtained on the basis of Timoshenko beam equations for an equivalent homogeneous beam, using the effective moduli suggested by Voigt and by Reuss, respectively. The influence of the damping coefficient and the load velocity on the beam response are analyzed in some detail. The limiting case for no damping and the resonance behavior of the steady-state solutions are also investigated.

Nomenclature

$a_1, a_2, \ldots, a_{13}$	= coefficients of the composite beam Eqs. (51-63) defined in Ref. 13
A	= total cross-sectional area
A_1, A_2	= cross-sectional areas of stiff and soft layers, respectively
b	= width of the composite beam
d_1, d_2	= thicknesses of the stiff and soft layers, respectively
E	= effective Young's modulus
E_1, E_2	= Young's moduli of stiff and soft layers, respectively
F_0	= magnitude of the concentrated force
F_0^*	= nondimensional force parameter $(F_0/a_1\xi)$
G	= effective shear modulus
G_1, G_2	= shear moduli of stiff and soft layers, respectively
h	= total depth of the composite beam
I	= total amount of inertia
I_1, I_2	= moment of inertia of stiff and soft layers, respectively
k	= spring constant (foundation modulus)
$p(x,t)$	= distributed lateral load per unit span
q	= is
r	= $x - vt$
R	= dimensionless length (r/r_1)
r_1, r_2	= radius of gyration of stiff and soft layers, respectively
s	= Fourier transform parameter
t	= time
v	= velocity of the moving load
v_0	= $\sqrt{G_2/\rho_2}$
c_s	= $\sqrt{\kappa_2} v_0$, shear velocity of the soft layer
w	= transverse displacement of composite beam
W	= dimensionless transverse displacement (w/r_1)
$\bar{W}$	= exponential Fourier transform of W
x	= distance along the beam
β_0	= damping coefficient
ϕ	= local (micro) rotation of the stiff layer
ψ	= gross (macro) rotation
$\bar{\phi}$	= exponential Fourier transform of ϕ
$\bar{\psi}$	= exponential Fourier transform of ψ
ρ	= effective mass density
κ_1, κ_2	= shear correction coefficients for stiff and soft layers, respectively
ν_1, ν_2	= Poisson's ratios of stiff and soft layers, respectively
ρ_1, ρ_2	= mass densities of stiff and soft layers, respectively
$\eta, \xi, \gamma, \alpha, \beta, \theta$	= nondimensional parameters given by Eq. (8)
$\delta(\ )$	= Dirac delta function
$H(\ )$	= Heaviside unit function

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Introduction

In recent years considerable attention has been paid to the problem of beams subjected to moving loads.[1-9] Simultaneously, the importance of structural elements fabricated from composite materials has increased significantly and, as a result, a variety of mathematical models have been proposed to describe such structural elements.[10-13] One such continuum model, which includes microstructure for a laminated beam, has been advanced by Sun.[13]

In the present investigation, Sun's theory forms the basis for studying the steady-state response of a composite beam resting on an elastic foundation with or without damping and subjected to a moving load of constant magnitude. The material constants and geometrical characteristics of the composite beam used here are the same as in Ref. 13. The beam consists of five stiff layers and four soft layers. The notions of steady-state have also been used by many authors in the past[6] and more recently[4,7] for studying the moving load problems on a flexibly supported Timoshenko beam with and without damping and predicting the behavior of shells to moving loads. These references are not complete but merely indicative. Transient behavior of a semi-infinite Timoshenko beam has also been investigated in the past by Boley[8] for suddenly applied and gradually varying impact loads and more recently by Florence[9] for a moving concentrated load. The latter problem, like the earlier steady-state solution[4] also has a singular behavior when the load speeds equal either the shear or the bar velocity. Thus, it is to be noted that, although the steady-state[4] and the short transient solution[9] for a moving concentrated load both predict singular behavior at the so-called "critical speeds," they nevertheless approach the step-load asymptotic solution,[5] except for the above-critical velocities of the load.

Results are also obtained using the effective modulus theories suggested by Voigt[14] and by Reuss.[15] The effective moduli of the representative homogeneous beams are used in the Timoshenko beam equations. The damping and the elastic modulus of the foundation are introduced as usual. The results are compared with those based on the microstructure theory. It is found that the deformation pattern predicted by the microstructure beam theory is quite different from that calculated on the basis of either the Voigt or the Reuss theory.

Analysis Based on Microstructure Theory

The microstructure beam theory advanced by Sun[13] yields a system of displacement equations of motion in three dependent variables w, ϕ, and ψ as

$$a_1\frac{\partial^2 w}{\partial x^2}-a_2\frac{\partial\psi}{\partial x}-a_3\frac{\partial\phi}{\partial x}+\frac{p}{\xi}=a_4\frac{\partial^2 w}{\partial t^2} \tag{1}$$

$$a_2\frac{\partial w}{\partial x}+a_5\frac{\partial^2\psi}{\partial x^2}-a_6\psi-a_7\frac{\partial^2\phi}{\partial x^2}+a_8\phi=a_9\frac{\partial^2\psi}{\partial t^2}-a_{10}\frac{\partial^2\phi}{\partial t^2} \tag{2}$$

$$a_3\frac{\partial w}{\partial x}-a_7\frac{\partial^2\psi}{\partial x^2}+a_8\psi+a_{11}\frac{\partial^2\phi}{\partial x^2}-a_{12}\phi=-a_{10}\frac{\partial^2\psi}{\partial t^2}+a_{13}\frac{\partial^2\phi}{\partial t^2} \tag{3}$$

All quantities entering these equations are defined in Ref. 13.

If a load of constant magnitude F_0 moves with constant velocity v over the composite beam, which is supported by springs placed parallel to dashpots, the quantity p in Eq. (1) becomes

$$p(x,t)=F_0\delta(x-vt)-kw-\beta_0(\partial w/\partial t) \tag{4}$$

Stipulating a steady-state solution and introducing r as the independent variable, one can write

$$(a_1-a_4v^2)\frac{d^2w}{\partial r^2}-a_2\frac{d\psi}{dr}-a_3\frac{d\phi}{dr}-\frac{1}{\xi}(kw-\beta_0v\frac{dw}{dr})=-\frac{F_0}{\xi}\delta(r) \tag{5}$$

$$a_2\frac{dw}{dr}+(a_5-a_9v^2)\frac{d^2\psi}{dr^2}-a_6\psi-(a_7-a_{10}v^2)\frac{d^2\phi}{dr^2}+a_8\phi=0 \tag{6}$$

$$a_3\frac{dw}{dr}-(a_7-a_{10}v^2)\frac{d^2\psi}{dr^2}+a_8\psi+(a_{11}-a_{13}v^2)\frac{d^2\phi}{dr^2}-a_{12}\phi=0 \tag{7}$$

The solution may now be constructed with the use of Fourier transforms. Introducing the nondimensional variables

$$W=\frac{w}{r_1},\quad R=\frac{r}{r_1},\quad v_0=\sqrt{G_2/\rho_2},\quad F^*=\frac{F_0}{a_1\xi} \tag{8a}$$

$$\alpha=\frac{kr_1^2}{a_1\xi},\quad \beta=\frac{\beta_0r_1v_0}{2\xi a_1},\quad \gamma=\frac{G_1}{G_2},\quad \lambda=\frac{\rho_1}{\rho_2} \tag{8b}$$

$$\theta=\frac{v}{v_0},\quad \eta=\frac{d_1}{d_1+d_2},\quad \xi=\frac{h}{d_1+d_2} \tag{8c}$$

and the nondimensional constants

$$\eta_1=\frac{a_1}{a_2},\quad \eta_2=\frac{a_5}{a_1r_1^2},\quad \eta_3=\frac{a_7}{a_1r_1^2},\quad \eta_4=\frac{a_{11}}{a_1r_1^2} \tag{9}$$

$$\alpha_1=\frac{a_1}{a_4v_0^2},\quad \alpha_2=\frac{a_5}{a_4v_0^2},\quad \alpha_3=\frac{a_7}{a_{10}v_0^2},\quad \alpha_4=\frac{a_{11}}{a_{13}v_0^2} \tag{10}$$

Eqs. (5-7) are transformed using the following pair

$$\bar{u}(s)=\int_{-\infty}^{\infty}u(R)e^{-isR}dR \tag{11a}$$

$$u(R)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{u}(s)e^{isR}ds \tag{11b}$$

Further simplifications are achieved using the relations deduced in the Appendix. The results are

$$\bar{W}=\frac{F^*(s^4p_4+s^2p_2+1)}{\Delta} \tag{12}$$

$$\bar{\psi}=\frac{F^*is(s^2p_5+1)}{\Delta} \tag{13}$$

$$\bar{\phi}=\frac{F^*is(s^2p_6+1)}{\Delta} \tag{14}$$

where

$$\Delta=(s^2p_1-2\beta\theta is+\alpha)(s^4p_4+s^2p_2+1)-s^2(s^2p_3+1) \tag{15}$$

The coefficients $p_1, p_2,..., p_6$ in Eqs. (12-15) depend upon the velocity parameter θ and can be expressed in the foregoing nondimensional quantities in Eqs. (8-10). These are

$$p_1=(1-\theta^2/\alpha_1) \tag{16}$$

$$p_2=\left[\eta_4(1-\theta^2/\alpha_4)-\eta_2(\eta\eta_1-2\eta-\eta_1+1)(1-\theta^2/\alpha_2)-2\eta\eta_3(1-\theta^2/\alpha_3)\right]\Big/\left[1-(1-\eta)/\eta_1\right] \tag{17}$$

$$p_3=\left[\eta_4(1-\theta^2/\alpha_4)+2\eta_3(\eta_1-1)(1-\theta^2/\alpha_3)+\eta_2(\eta_1-1)^2(1-\theta^2/\alpha_2)\right]\Big/\left[\eta_1/(1-\eta)-1\right] \tag{18}$$

$$p_4=\frac{\eta_1^2[\eta_2\eta_4(1-\theta^2/\alpha_2)(1-\theta^2/\alpha_4)-\eta_3^2(1-\theta^2/\alpha_3)^2]}{\eta_1/(1-\eta)-1} \tag{19}$$

$$p_5=\frac{\eta_1\eta_4(1-\theta^2/\alpha_4)+\eta_1\eta_3(\eta_1-1)(1-\theta^2/\alpha_3)}{\eta_1/(1-\eta)-1} \tag{20}$$

$$p_6=\frac{\eta_1\eta_2(\eta_1-1)(1-\theta^2/\alpha_2)+\eta_1\eta_3(1-\theta^2/\alpha_3)}{\eta_1/(1-\eta)-1} \tag{21}$$

The nondimensional constants η_i, α_i; for $i=1, 4$ defined in expressions (9) and (10) and used in Eqs. (16-21) can be expressed in terms of geometrical and material properties of the composite beam. The results are summarized in the Appendix.

The denominator Δ of Eqs. (12-14) is a sixth-order polynomial in s with real and imaginary coefficients. Once the roots of the characteristic equation $\Delta=0$ are determined, the transformed functions $\bar{W}(s)$, $\bar{\psi}(s)$, and $\bar{\phi}(s)$ can be expanded into partial fractions; each term of which can be set in a general form of the type $A/(s+a+ib)$, where A is a constant and $-(a+ib)$ is a root in which a and b can be positive, negative, or even zero.

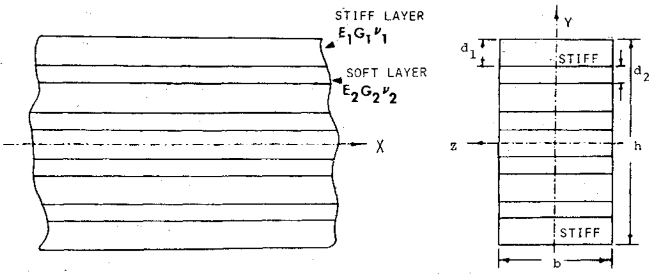

Fig. 1 Layout of composite beam.

The inverse transform of $\bar{W}(s)$, $\bar{\psi}(s)$, and $\bar{\phi}(s)$ can be accomplished by recalling that

$$F^{-1}\frac{1}{s+a+ib} = -\operatorname{sgn}(b)\, i e^{bR} e^{-iaR} H[-R \operatorname{sgn}(b)] \tag{22}$$

where F^{-1} represents inverse Fourier transform, and $H[R]$ and $\operatorname{sgn}(b)$ are generalized functions.

Characteristic Equation

The characteristic equation [Eq. (15)] can be rewritten as

$$q^6 p_1 p_4 + 2\beta\theta p_4 q^5 + (p_3 - p_1 p_2 - \alpha p_4) q^4 - 2\beta\theta p_2 q^3 + (p_1 + \alpha p_2 - 1) q^2 + 2\beta\theta q - \alpha = 0 \tag{23}$$

where a substitution, $q = is$, is introduced. As β is associated with only odd powers of q, it is evident by inspection that for nonzero β the roots cannot be purely imaginary. This means $\bar{W}(s)$, $\bar{\psi}(s)$, and $\bar{\phi}(s)$ can have poles anywhere on the complex s plane except on its real axis. This in general, however, is not true for $\beta = 0$. The types of roots associated with Eq. (23) for the cases with and without damping are briefly discussed in the following.

Absence of Damping

The characteristic equation is

$$s^6 p_1 p_4 + s^4 (p_4\alpha + p_1 p_2 - p_3) + s^2 (p_1 + p_2\alpha - 1) + \alpha = 0 \tag{24}$$

From the analysis similar to one in Ref. 16, the discriminant Ω of this equation can be shown to be

Table 1 Types of roots of the characteristic equation [Eq. (24)]

Type	General form of the root	Limiting forms
I	$\pm(a_1 \pm ib_1),\ \pm ib_2$	$a_1 = 0$
II	$\pm ib_1,\ \pm ib_2,\ \pm ib_3$	$b_1 = b_2$ or $b_1 = b_2 = b_3$
III	$\pm a_1,\ \pm a_2,\ \pm ib_3$	$a_1 = a_2$

$$\Omega = p_1^2 p_4^2 \alpha^2 - \frac{2}{3} p_1 p_4 \alpha (p_4\alpha + p_1 p_2 - p_3)(p_1 + p_2\alpha - 1) + \frac{4}{27} p_1 p_4 (p_1 + p_2\alpha - 1)^3 + \frac{4}{27}\alpha (p_4\alpha + p_1 p_2 - p_3)^3 - \frac{1}{27}(p_4\alpha + p_1 p_2 - p_3)^2 (p_1 + p_2\alpha - 1)^2 \tag{25}$$

Further, utilizing the sign of Ω and of the coefficients in Eq. (24), as discussed in Ref. 17, one can establish that the following three distinct types of roots exist for the characteristic equation. The roots can, however, assume either of these forms depending upon the location of the point in the $(\theta - \alpha)$ plane. Without any loss of generality α can be assumed positive.

The following parameters are used in plotting the curve of $\Omega = 0$ shown in Fig. 9, which bounds the areas of the first quadrant of the $(\theta - \alpha)$ plane into regions in which the characteristic equation [Eq. (24)] possesses a definite type of root. Those correspond to a composite beam consisting of five stiff layers and four soft layers sketched in Fig. 1.

$$\gamma = 100, \quad \lambda = 2, \quad \eta = 0.8, \quad \xi = 4.8, \quad \nu_1 = 0.2, \quad \nu_2 = 0.35 \tag{26}$$

These parameters are the same as originally used in Ref. 13 for plotting the dispersion curves for the flexural waves. The solutions for displacement $w(R)$ are evaluated for each of the cases shown in Table 1 with the successive use of Eq. (22). The results are summarized in Table 2.

For case III, the integrand of the inversion integral had four simple poles on the real axis, and it was difficult to ascertain which one of the line contours around the poles should be passed in the upper half plane and which should be passed in the lower half. This ambiguity is avoided by considering the present case as a limiting case of the beam on the damped foundation.

Table 2 Solution for $W(R)/F^*$ in the absence of damping

Case	
I	$\dfrac{1}{2p_1p_4[(b_2+a_1^2-b_1^2)^2+4a_1^2b_1^2]}\left[\dfrac{n_3\cos a_1\lvert R\rvert - n_4\sin a_1\lvert R\rvert}{a_1^2+b_1^2}\cdot\dfrac{e^{-b_1\lvert R\rvert}}{2a_1b_1}+\dfrac{p_4b_2^4-p_2b_2^2+1}{b_2}\cdot e^{-b_2\lvert R\rvert}\right]$
II	$\dfrac{1}{2p_1p_4}\sum_{j=1}^{j=3}\dfrac{\ell_j e^{-b_j\lvert R\rvert}}{b_j}$; where $\ell_j = \dfrac{b_j^4p_4 - b_j^2p_2 + 1}{\prod_{k=1,3;\ k\neq j}(b_j^2-b_k^2)}$
III	$\dfrac{1}{2p_1p_4}\left[\left\{m_3\dfrac{e^{-b_3R}}{b_3} - 2m_1\dfrac{\sin a_1R}{a_1}\right\}H(R) + \left\{m_3\dfrac{e^{b_3R}}{b_3} + 2m_2\dfrac{\sin a_2R}{a_2}\right\}H(-R)\right]$

where

$$n_1 = p_4(a_1^4 + b_1^4 - 6a_1^2b_1^2) + p_2(a_1^2 - b_1^2) + 1; \quad n_3 = n_1a_1(b_2^2 + a_1^2 - 3b_1^2) + n_2b_1(3a_1^2 - b_1^2 + b_2^2)$$

$$n_2 = 4p_4a_1b_1(a_1^2 - b_1^2) + 2p_2a_1b_1; \quad n_4 = n_2a_1(b_2^2 + a_1^2 - 3b_1^2) - n_1b_1(3a_1^2 - b_1^2 + b_2^2)$$

$$m_1 = \frac{a_1^4p_4 + a_1^2p_2 + 1}{(a_1^2 - a_2^2)(a_1^2 + b_3^2)}; \quad m_2 = \frac{a_2^4p_4 + a_2^2p_2 + 1}{(a_2^2 - a_1^2)(a_2^2 + b_3^2)}; \quad m_3 = \frac{b_3^4p_4 - b_3^2p_2 + 1}{(a_1^2 + b_3^2)(a_2^2 + b_3^2)}$$

Table 3 Types of roots of Eq. (23)

Type	Roots in q plane	Roots in s plane
I	$-a_1 \pm ib_1, a_2 \pm ib_2, \pm a_3$	$\pm b_1 + ia_1, \pm b_2 - ia_2, \pm ia_3$
II	$-a_1 \pm ib_1, \pm a_2, a_3, a_4$	$\pm b_1 + ia_1, \pm ia_2, -ia_3, -ia_4$
III	$a_1 \pm ib_1, \pm a_2, -a_3, -a_4$	$\pm b_1 - ia_1, \pm ia_2, ia_3 ia_4$
IV	$-a_1, -a_2, -a_3, a_4, a_5, a_6$	$ia_1, ia_2, ia_3, -ia_4, -ia_5, -ia_6$

Presence of Damping

The distinct types of roots associated with Eq. (23) in the presence of damping are determined to have one or another of the forms shown in Table 3.

The procedure for obtaining the inversion of Eqs. (12-14) is quite similar to the one determined for the case without damping. The algebraic details are omitted here. Results corresponding to each one of the cases (I-IV) are summarized in Table 4. The constants used in Table 4 are

$$m_5 = p_4[(b_1^2 - a_1^2)^2 - 4b_1^2 a_1^2] + p_2(b_1^2 - a_1^2) + 1$$

$$m_6 = 2b_1 a_1 [2p_4(b_1^2 - a_1^2) + p_2]$$

$$m_7 = [b_1^2 - b_2^2 - (a_1 + a_2)^2][b_1^2 - a_1^2 + a_3^2] - 4b_1^2 a_1 (a_1 + a_2)$$

$$m_8 = 2b_1(a_1 + a_2)(b_1^2 - a_1^2 + a_3^2) + 2a_1 b_1 [b_1^2 - b_2^2 - (a_1 + a_2)^2]$$

$$m_9 = \frac{a_3^4 p_4 - a_3^2 p_2 + 1}{[b_1^2 + (a_3 - a_1)^2][b_2^2 + (a_3 + a_2)^2]}$$

$$n_5 = p_4[(b_2^2 - a_2^2)^2 - 4b_2^2 a_2^2] + p_2(b_2^2 - a_2^2) + 1$$

$$n_6 = 2b_2 a_2 [2p_4(b_2^2 - a_2^2) + p_2]$$

$$n_7 = [b_1^2 - b_2^2 + (a_1 + a_2)^2][b_2^2 - a_2^2 + a_3^2] + 4b_2^2 a_2 (a_1 + a_2)$$

$$n_8 = -2b_2(a_1 + a_2)(b_2^2 - a_2^2 + a_3^2) + 2a_2 b_2 [b_1^2 - b_2^2 + (a_1 + a_2)^2]$$

$$n_9 = \frac{a_3^4 p_4 - a_3^2 p_2 + 1}{[b_1^2 + (a_3 + a_1)^2][b_2^2 + (a_3 - a_2)^2]}$$

$$n_{10} = (b_1^2 - a_1^2 + a_2^2)[b_1^2 - (a_1 + a_3)(a_1 + a_4)] - 2a_1 b_1^2 (2a_1 + a_3 + a_4)$$

$$n_{11} = b_1(2a_1 + a_3 + a_4)(b_1^2 - a_1^2 + a_2^2) + 2a_1 b_1 [b_1^2 - (a_1 + a_3)(a_1 + a_4)]$$

$$\ell_5 = \frac{a_2^4 p_4 - a_2^2 p_2 + 1}{b_1^2 + (a_1 - a_2)^2}$$

$$\ell_j = \frac{a_j^4 p_4 - a_j^2 p_2 + 1}{b_1^2 + (a_1 + a_j)^2}; \quad j = 6,\ 7,\ \text{and}\ 8$$

Analysis Based on Effective Modulus Theories

A periodically layered composite beam of two elastic materials, Fig. 1, can also be considered as a homogeneous beam without microstructure with its effective moduli so determined that it predicts the gross or net behavior of the composite beam to a certain approximation. Two such basic

Table 4 Solution for $W(R)/F^*$ in the presence of damping

Case	In the position ahead of the load; $R>0$
I	$\dfrac{1}{p_1 p_4}\left[\dfrac{-e^{-a_1 R}}{b_1}\dfrac{\{(m_5 m_7 + m_6 m_8)\sin b_1 R + (m_6 m_7 - m_5 m_8)\cos b_1 R\}}{(m_7^2 + m_8^2)} + \dfrac{e^{-a_3 R}}{2a_3}\cdot m_9\right]$
II	$\dfrac{1}{p_1 p_4}\left[\dfrac{-e^{-a_1 R}}{b_1}\dfrac{\{(m_5 n_{10} + m_6 n_{11})\sin b_1 R + (m_6 n_{10} - m_5 n_{11})\cos b_1 R\}}{(n_{10}^2 + n_{11}^2)} + \dfrac{e^{-a_2 R}\ell_5}{2a_2(a_2 + a_3)(a_2 + a_4)}\right]$
III	Expression same as for $R<0$ in case II
IV	$\dfrac{1}{p_1 p_4}\sum_{j=1}^{j=3}\dfrac{(a_j^4 p_4 - a_j^2 p_2 + 1)e^{-a_j R}}{[\prod_{k=4,6}(a_j + a_k)][\prod_{k=1,3;\ j\neq k}(a_j - a_k)]}$
Case	**In the position behind the load; $R<0$**
I	$\dfrac{1}{p_1 p_4}\left[\dfrac{e^{-a_2\lvert R\rvert}}{b_2}\dfrac{\{(n_5 n_7 + n_6 n_8)\sin b_2 \lvert R\rvert + (n_6 n_7 - n_5 n_8)\cos b_2 R\}}{(n_7^2 + n_8^2)} + \dfrac{e^{-a_3\lvert R\rvert}}{2a_3}\cdot n_9\right]$
II	$\dfrac{1}{p_1 p_4}\left[\dfrac{\ell_6 e^{-a_2\lvert R\rvert}}{2a_2(a_3 - a_2)(a_4 - a_2)} - \dfrac{1}{(a_3 - a_4)}\left\{\dfrac{\ell_7 e^{-a_3\lvert R\rvert}}{(a_2^2 - a_3^2)} - \dfrac{\ell_8 e^{-a_4\lvert R\rvert}}{(a_2^2 - a_4^2)}\right\}\right]$
III	Expression same as for $R>0$ in case II
IV	$\dfrac{1}{p_1 p_4}\sum_{j=4}^{j=6}\dfrac{(a_j^4 p_4 - a_j^2 p_2 + 1)e^{-a_j\lvert R\rvert}}{[\prod_{k=1,3}(a_j + a_k)][\prod_{k=4,6;\ j\neq k}(a_j - a_k)]}$

models have been advanced by Voigt[14] and Reuss.[15] The effective Young's and shear moduli of the composite system in each of the preceding two cases can be expressed in terms of their constituents as

$$E_v = \eta E_1 + (1-\eta)E_2, \quad G_v = \eta G_1 + (1-\eta)G_2 \tag{27}$$

using Voigt's assumption and

$$\frac{1}{E_R} = \frac{\eta}{E_1} + \frac{1-\eta}{E_2}, \quad \frac{1}{G_R} = \frac{\eta}{G_1} + \frac{1-\eta}{G_2} \tag{28}$$

on the basis of Reuss' assumption. The effective mass density ρ, however, remains the same for both models. It is given by

$$\rho = \eta\rho_1 + (1-\eta)\rho_2 \tag{29}$$

We, thus, obtained two equivalent homogeneous beams, replacing the original composite beam. Timoshenko beam equations can now be used in conjunction with Eqs. (27-29) to analyze the response of the composite beam without microstructure. Using the earlier concept of the steady-state solution with $r = x - vt$ and an identical external force distribution, the problem of the equivalent composite beam supported by an elastic foundation, can be formulated as

$$(e_2^2 - \theta^2)\frac{d^2\psi}{dR^2} - e_1^2 b_1 \psi + e_1^2 b_1 \frac{dW}{dR} = 0 \tag{30}$$

$$(e_1^2 - \theta^2)\frac{d^2 W}{dR^2} + 2b_2\theta\beta\frac{dW}{dR} - b_2\alpha W - e_1^2\frac{d\psi}{dR} = -b_2 F^* \delta(R) \tag{31}$$

where the following additional nondimensional constants are used

$$b_1 = \frac{Ar_1^2}{I}, \quad b_2 = \frac{a_1\xi}{A\rho v_0^2}, \quad e_1^2 = \frac{\kappa G}{\rho v_0^2}, \quad e_2^2 = \frac{E}{\rho}\frac{1}{v_0^2} \tag{32}$$

It is found[18] that for a homogeneous Timoshenko beam with a rectangular cross section, the shear correction coefficient k assumes a value of 0.822. Analysis based on recent investigation[19] suggested a value for static shear correction factor as 0.81 for the equivalent composite beam. There were indications, however, that the k factors calculated in this manner, may not yield the closest approximation to the correct k factor determined for laminates using dynamic consideration.[20] Thus, a value of 0.822 is used in the present consideration.

The foregoing constants b_i and e_i can be expressed in terms of the mass density ratio λ, shear modulus ratio γ, and Poisson's ratios ν_1, ν_2 of the constituent layers. The expressions for these quantities are shown here for both Voigt and Reuss models. Defining the ratio

$$e_2^2/e_1^2 = \nu_0^2 \tag{33}$$

we have

1) For the Voigt model

$$\nu_0^2 = \frac{2\eta(1+\nu_1)\gamma + 2(1-\eta)(1+\nu_2)}{\kappa[1-\eta+\eta\gamma]} \tag{34a}$$

$$e_1^2 = \frac{\kappa[1-\eta+\eta\gamma]}{1-\eta+\eta\lambda} \tag{34b}$$

2) For the Reuss model

$$\nu_0^2 = \frac{2}{\kappa}\left[\frac{\eta + (1-\eta)\gamma}{\eta/(1+\nu_1) + (1-\eta)\gamma/(1+\nu_2)}\right] \tag{35a}$$

$$e_1^2 = \frac{\kappa\gamma}{[(1-\eta)\gamma+\eta][(1-\eta)+\eta\lambda]} \tag{35b}$$

b_1, b_2 having been found the same for both the Voigt and the Reuss models, are given as

$$b_1 = \eta^2/\xi^2, \qquad b_2 = \frac{\kappa(1-\eta+\eta\gamma)}{1-\eta+\eta\lambda} \tag{36}$$

With the use of the Fourier transform defined in Eq. (11), Eqs. (30) and (31) can be solved for $\bar{W}/F^*$ and $\bar{\psi}/F^*$. The results are

$$\bar{W}/F^* = -b_2[(e_2^2-\theta^2)s^2 + e_1^2 b_1]/\Delta_1 \tag{37}$$

$$\bar{\psi}/F^* = -e_1^2 b_1 b_2 is/\Delta_1 \tag{38}$$

in which

$$\Delta_1 = -(e_2^2-\theta^2)(e_1^2-\theta^2)s^4 + 2b_2\theta\beta is^3(e_2^2-\theta^2) + s^2\{e_1^2 b_1\theta^2 - b_2\alpha(e_2^2-\theta^2)\} + 2b_1 b_2\theta\beta ise_1^2 - e_1^2 b_1 b_2\alpha \tag{39}$$

The types of roots and the deformation patterns can be extracted from Eqs. (37-39) in a manner similar to the one

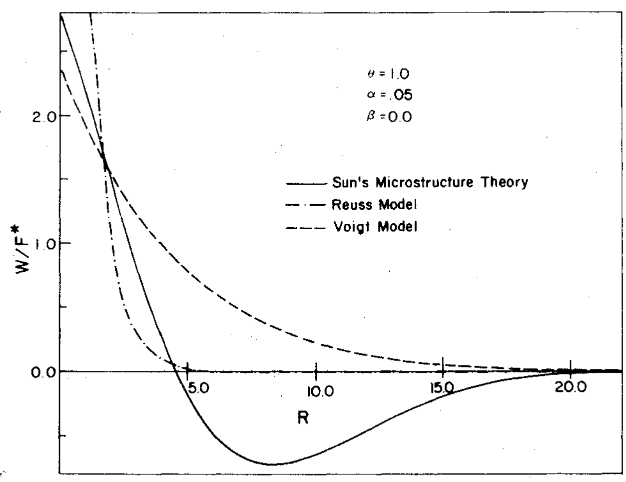

Fig. 2 Comparison of predicted response without damping.

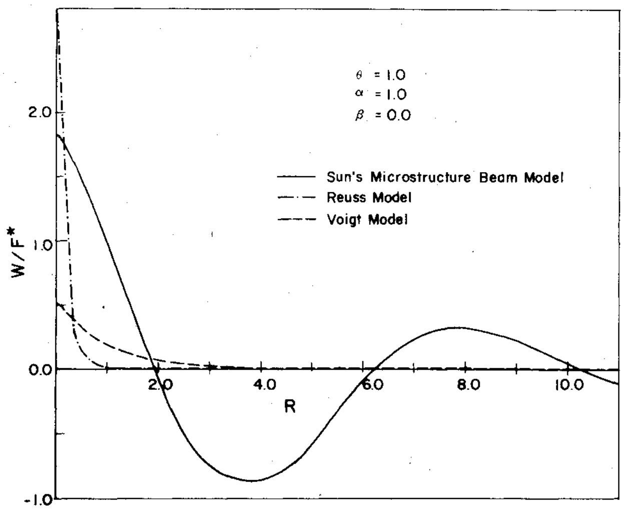

Fig. 3 Comparison of predicted response without damping.

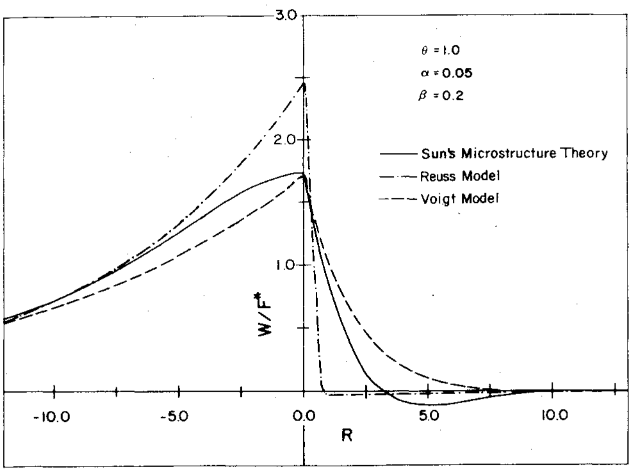

Fig. 4 Comparison of predicted response with damping.

outlined in the previous sections. The final results are, however, presented for these models in the form of graphs (Figs. 2, 3, and 4) for clarity of comparison.

Numerical Results

In each of Figs. 2-4 for a load velocity parameter θ of unit magnitude and an arbitrarily chosen value of α, three deflection curves are drawn. One corresponds to Sun's microstructure theory,[13] and the other two correspond to the effective modulus theory of Voigt[14] and of Reuss,[15] respectively. The deflections, in the undamped case $\beta=0$, which are symmetrical about the point of application of the load, are shown in Figs. 2 and 3. The results of the microstructure theory show a gradual variation in deformation pattern from the point of its maximum positive value under the load to a point of its maximum negative value some distance away from the load and thereafter a subsequent oscillatory behavior. By contrast the deflection curves based on the Reuss and the Voigt models appear to be less realistic because of their large gradients.

The most striking feature of the case which includes damping (Fig. 4) is the disappearance of the symmetry between the solutions ahead of and behind the load. The deformation in this case reaches its maximum either under the load or in the portions behind it, but decreases gradually away from its position on either side.

All the remaining results as presented in Figs. 5-9 are based on the microstructure theory. The effect of elastic foundation is to produce an oscillating behavior of the composite beam

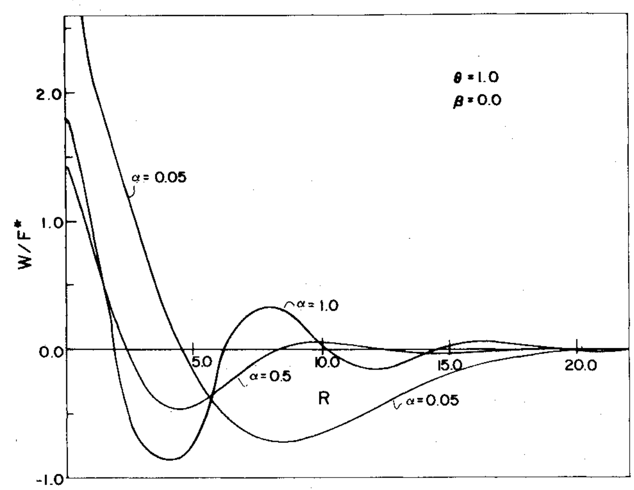

Fig. 5 Results showing the influence of elastic foundation in the absence of damping.

deformations. For an undamped beam such an effect is more pronounced and can be observed graphically from Fig. 5, wherein deformation curves are plotted for three representative values of elastic foundation moduli.

For a damped beam, such an oscillatory behavior is found to be reduced, even in the presence of a fairly small amount of damping (Fig. 6). The magnitude of deformation is larger for a soft foundation than for the case when the beam is placed on a relatively stiff foundation.

Figure 7 shows the effect of varying load velocity. The nearly symmetrical form of deformation is retained for low velocity of the moving load ($\theta \leq 0.2$), but for unit velocity

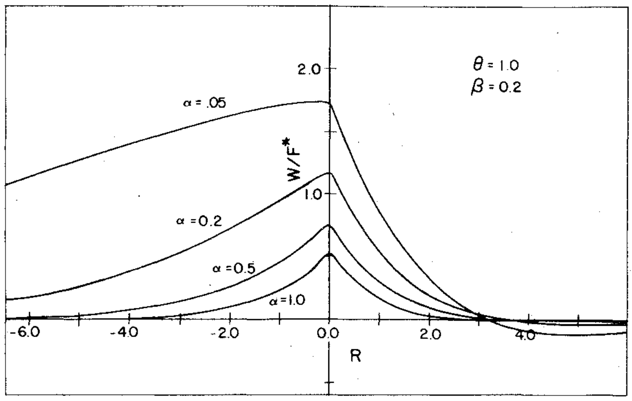

Fig. 6 Results showing the influence of elastic foundation in the presence of damping.

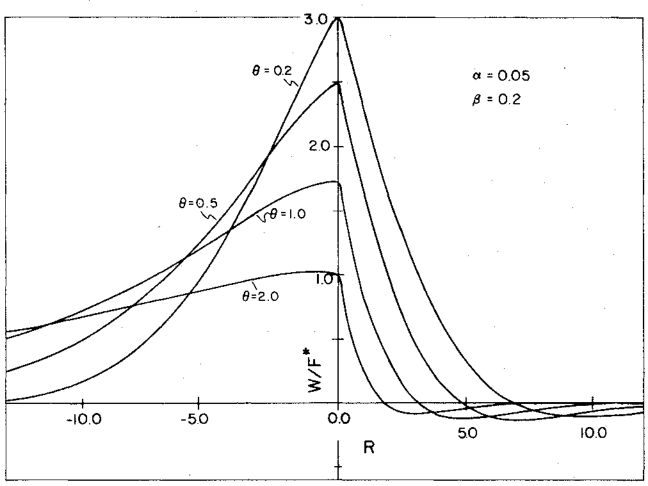

Fig. 7 Results showing the influence of load velocities.

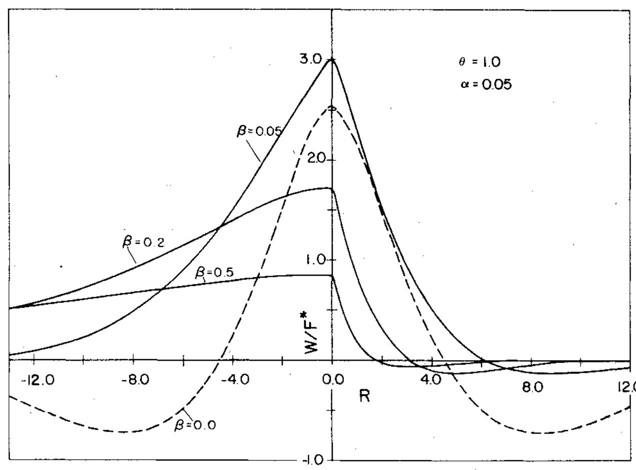

Fig. 8 Results showing influence of damping.

($\theta = 1$) the pattern in the portions behind the load is determined to have a shallower profile than its shape appearing ahead of the load.

Figure 8 shows the influence of damping on the beam response plotted for $\theta = 1.0$ and $\alpha = 0.05$. The curve for no damping is shown by broken lines. It is observed that for a small value of $\beta (\beta \leq 0.05)$ the nature of variation ahead of and behind the load is nearly symmetrical. For a large value of damping such variations are not possible; the deformation pattern behind the load tends to acquire a shallow shape whereas, in the portions ahead of the load, it has a drooping behavior.

For a beam which is supported by a stiff foundation and subjected to a fast moving load ($\theta \geq 5$), the degree of shallowness is effectively reduced behind the load. The reason is the change in sign and form of the roots of the characteristic equation resulting in a solution which is a combination of damped oscillatory wave and exponential decay.

Determination of Resonant Speeds

The Fourier transforms of ω, ψ, and ϕ, as given by Eqs. (12-14) in its most general form with β and α terms included, can be written as

$$\frac{\bar{W}}{F^*} = -\frac{q^4 p_4 - q^2 p_2 + 1}{\Delta} \tag{40}$$

$$\frac{\bar{\psi}}{F^*} = \frac{-q[1 - q^2 p_5]}{\Delta} \tag{41}$$

and

$$\frac{\bar{\phi}}{F^*} = \frac{-q[1 - q^2 p_6]}{\Delta} \tag{42}$$

where

$$\Delta = q^6 p_1 p_4 + 2\beta\theta p_4 q^5 + (p_3 - p_1 p_2 - \alpha p_4) q^4 - 2\beta\theta p_2 q^3 + (p_1 + \alpha p_2 - 1) q^2 + 2\beta\theta q - \alpha \tag{43}$$

A simple substitution $q = is$ is introduced into the foregoing expressions.

Following an analysis similar to Ref. 4, one may anticipate that the load velocities which lead to either p_1 or p_4 equal to zero, may represent critical load velocities, meaning, thereby, that for such a speed the response becomes unbounded. It is however noted that such a speed, if it exists, reduces the order of the original differential equations. The resulting transformed solutions look like: if $p_1 = 0$, as

$$\bar{W}/F^* = \left[-(q^4 p_4 - q^2 p_2 + 1)\right] \Big/ \left[2\beta\theta p_4 q^5 + (p_3 - \alpha p_4) q^4 - 2\beta\theta p_2 q^3 + (\alpha p_2 - 1) q^2 + 2\beta\theta q - \alpha\right] \tag{44}$$

and if, $p_4 = 0$, as

$$\bar{W}/F^* = \left[-(1 - q^2 p_2)\right] \Big/ \left[(p_3 - p_1 p_2) q^4 - 2\beta\theta p_2 q^3 + (p_1 + \alpha p_2 - 1) q^2 + 2\beta\theta q - \alpha\right] \tag{45}$$

In both of these cases, it is possible to invert the equations and the deflection would still be finite. These speeds therefore do not seem to represent true critical speeds.

In the limiting case, when $\beta = 0$, the response of the composite beam, as it can be seen from Eqs. (44) and (45), would still be finite for $p_4 = 0$, but no longer bounded for $p_1 = 0$. This means that the load velocity corresponding to $p_1 = 0$ seems the only true critical one.

The result for V_c is,

$$V_c = \sqrt{\frac{\kappa_2 G_2}{\rho_2}} \cdot \left[\frac{1 - \eta + \eta\gamma}{1 - \eta + \eta\lambda}\right]^{1/2} \tag{46}$$

The first factor in Eq. (46) is the propagation velocity of the shear force disturbance for the soft layer of the composite beam. The second factor represents a correction factor for the composite beam which depends upon the relative thickness of the stiff and soft layer, their shear moduli and mass density ratios. For a composite beam consisting of five stiff layers and four soft layers with its geometrical and material properties as given in Eq. (26), θ is determined to be 6.05183. Such a critical velocity, as discussed in Ref. 5 and commented upon in the introduction of this paper, may be the result of using a concentrated moving load and requires further investigation.

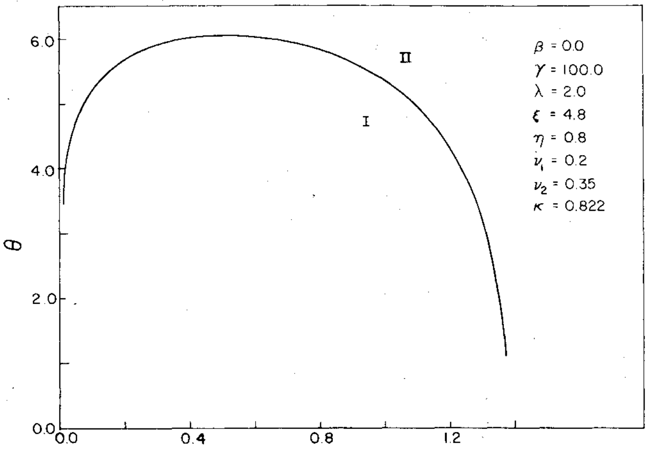

Fig. 9 Variation of critical velocities with the stiffness of elastic bed.

The other critical velocities for the composite beam occur for the cases when the characteristic equation (24) possesses double roots; $s = \pm a, \pm a, \pm ib$. This is a limiting form of case III mentioned in Table 1, which is in general of the type; $\pm a_1$, $\pm a_2$, and $\pm ib_3$.

For this particular case the inversion integrals are of the form

$$\int_{-\infty}^{\infty} \frac{dx}{(x-a)^2} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{dx}{(x+a)^2} \tag{47}$$

where x and a are real.

It is well known that an integral of the type in Eq. (47) does not exist even in the sense of a Cauchy principal value. It was discussed in Ref. 6 that such nonexistence of a solution to a physical problem implies a resonance, in the sense that the displacement becomes unbounded all along the beam. The value of the load velocity parameter θ, which yields an integral of the type in Eq. (47) defines the critical load velocities.

Imposing the condition for double roots and eliminating b, one can find, using Eq. (24), a condition in terms of θ, which is

$$(p_1 + p_2\alpha - 1)^2 - 3\alpha(p_4\alpha + p_1 p_2 - p_3) = 0 \tag{48}$$

where $p_1, p_2, \ldots, p_4$ are functions of only the load velocity parameter θ. They are defined in Eqs. (16-19). Values of θ for which Eq. (48) is satisfied, are the resonant speeds. In general there can be four roots to Eq. (48), but since only positive real roots are of physical significance, only those should be retained and the rest can be discarded.

The results for Ω (discriminant) $= 0$, in the absence of any damping are plotted in Fig. 9. Two regions are marked above and below the curve for $\Omega = 0$. It is found that for values of θ which lie in the region I of the $(\theta - \alpha)$ plane, the roots of the characteristic equation $\Delta = 0$, possess two pairs of complex-conjugate roots and two pure imaginary roots.

Along the curve for $\Omega=0$ (Fig. 9) as pointed out earlier, there are four real roots; two lie on the positive real axis and two on the negative real axis (the conjugate image points), all equal in magnitude. Further investigation reveals that when the points adjacent to the curve but somewhat away toward the region II are considered, two out of four real roots (one on either side of the origin) move toward and the remaining two move away from the origin, but never leave the real axis. Such a phenomenon has also been reported in the literature, for instance in Refs. 2 and 5, while studying the effect of moving load on Timoshenko beams or Euler-Bernoulli beams and merely confirms the consistency in behavior of such systems.

Conclusion

The microstructure theory for composite beams, as recently proposed by Sun, is found to predict a response to a moving constant load which is different from that predicted by either the Voigt or the Reuss effective modulus theories. As expected, the deflections of the beams are less oscillatory in the presence of damping than in its absence. Damping also reduces the magnitude of the deflection and shifts the position of maximum displacement from the point of application of the load to a point behind the load. The oscillatory behavior is more pronounced for a stiff elastic foundation than for a soft foundation. For the case when the load velocity is large, the soft foundation yields a deformation pattern which is more flat behind the load and steeper ahead of the load than when the foundation is relatively stiff.

Appendix

The following relations between the coefficients a_1, a_2, to a_{13} can be established:

$$a_3=(a_1-a_2) \tag{A1}$$

$$a_8=\eta a_2/(1-\eta) \tag{A2}$$

$$a_{12}=(a_1-a_2)+\eta a_2/(1-\eta) \tag{A3}$$

If we define for the sake of convenience a_0 as,

$$a_0=a_1a_2/(1-\eta)-a_2^2 \tag{A4}$$

then

$$a_6a_{12}-a_8^2=a_0 \tag{A5}$$

$$a_2a_{12}+a_3a_8=a_0 \tag{A6}$$

$$a_2^2a_{12}+2a_2a_3a_8+a_3^2a_6=a_1a_0 \tag{A7}$$

If we select a composite beam with its constituent layers having the same ratios of Young's modulus to their densities, i.e., if $E_1/E_2=\rho_1/\rho_2$; then it can be established tht

$$a_5/a_9=a_7/a_{10}=a_{11}/a_{13} \tag{A8}$$

Alternatively, a_0 from Eq. (A4) can be expressed as

$$a_0=(\kappa_1A_1G_1)(\kappa_2A_2G_2)/(1-\eta)^2 \tag{A9}$$

This is a positive quantity and will never be zero.

The nondimensional constants given by Eqs. (9) and (10) are evaluated in terms of material and geometrical parameters of the composite beam. The results are

$$\eta_1=(1-\eta)+\eta\gamma \tag{A10}$$

$$\eta_2=\frac{2(1+\nu_2)(1-\eta)(1+\xi^2)+2\eta\xi^2(1+\nu_1)\gamma}{\eta^2\eta_1\kappa} \tag{A11}$$

$$\eta_3=\frac{2(1+\nu_2)(1-\eta)}{\eta\eta_1\kappa} \tag{A12}$$

$$\eta_4=\frac{2(1+\nu_2)(1-\eta)+2\eta(1+\nu_1)\gamma}{\eta_1\kappa} \tag{A13}$$

$$\alpha_1=\frac{\eta_1\kappa}{1-\eta+\eta\lambda} \tag{A14}$$

$$\alpha_2=\frac{2(1+\nu_2)(1-\eta)(1+\xi^2)+2\eta(1+\nu_1)\gamma\xi^2}{(1-\eta)(1+\xi^2)+\eta\lambda\xi^2} \tag{A15}$$

$$\alpha_3=2(1+\nu_2) \tag{A16}$$

$$\alpha_4=\frac{2\eta(1+\nu_1)\gamma+2(1+\nu_2)(1-\eta)}{1-\eta+\eta\lambda} \tag{A17}$$

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